Communities of Organizations of Protest in the February 15, 2003, Anti-War Demonstrations in Eight Western Democracies

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Conceptual outline:

Premise 1: Attributes may parametrize relations and vertices (true for 2-mode networks).

Premise 2: For a graph G, a **community partition** is postulated to exist (at least in a tractable approximation).

Premise 3: Let $\{G^k\}_{k \in K}$ be a k-parametric family of graphs (all on the same set of vertices V). Then the corresponding parametric family of community partitions yields a weighted graph $\mathcal{G}^K$, which counts vertices interlocked in communities throughout K.

Premise 4: A community partition on the graph $\mathcal{G}^K$ is said to be composed of "**meta–communities**": Two vertices (of V) belong to a meta–community, whenever they both belong to a relatively higher number of communities of $\{G^k\}$ than vertices lying across different meta–communities.

Conceptual outline (cont.):

Premise 5: Let $\{G^{k,\ell}\}_{k\in K,\ell\in L}$ be a biparametric family of graphs (on V). For each ℓ, the graph $\mathcal{G}^{K,\ell}$ gives a ℓ-parametric family of partitions (to meta–communities), which can be aggregated (for all ℓ) to yield a weighted graph $\mathfrak{G}^{K,L}$, in which two vertices are connected if they are interlocked in the same k-meta–community, for some ℓ, and the weight is the number of such ℓ's.

Premise 6: A community partition on the graph $\mathfrak{G}^{K,L}$ is said to be composed of "**meta–meta–communities**": Two vertices belong to a meta–meta–community, whenever they both belong to a relatively higher number of meta–communities of $\{\mathcal{G}^{K,\ell}\}$ than vertices lying across different meta–meta–communities.

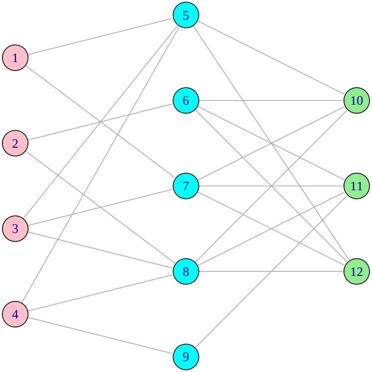
The International Peace Protest Survey (IPPS)

http://webh01.ua.ac.be/m2p/index.php?page=projects&page2=pproject&id=11

- On February 15, 2003, mass protests against an imminent war on Iraq took place throughout the world.
- More than seven million people in more than 300 cities all over the world had participated.
- The largest peace protests since the Vietnam War on one single day.
- An international team of social movement scholars set up the IPPS Project Survey (2003-4), coordinated by Stefaan Walgrave, to study this international protest event.
- Over 10,000 questionnaires were distributed in 8 countries during the demonstrations: in the UK, Italy, the Netherlands, Switzerland, USA, Spain, Germany and Belgium.
- About 6,000 completed questionnaires have been sent back, with a successful response rate of well above 50%.

Social Network Analysis of the IPPS Data

Here, we have decoded the survey data so that, for each of the 8 countries, we obtained a partial tripartite graph $G(A, B, C)$ of the following form:



In the IPPS data:

- B (**blue nodes**) is the population of respondents (varies in each country),
- A (**red nodes**) is a set of 16 (types of) organizations, to which respondents declared affiliation,
- C (**green nodes**) is a set of 10 attitudes with regard to the meaning of the war, about which respondents expressed their positions, opinions etc.

The 16 **organizations** to which respondents disclosed their affiliation (as many as they wanted to pick up) are:

1. Church
2. Anti-Racist
3. Student
4. Labor Union – Professional
5. Political Party
6. Women
7. Sports – Recreation
8. Environmental
9. Art, Music & Education
10. Neighborhood
11. Charitable
12. Anti-Globalist
13. Third World
14. Human Rights
15. Peace
16. Other

The 10 **attitudes** with regard to the war that respondents expressed (by answering with a Yes or No or NA the corresponding question) are:

1. USA Crusade against Islam
2. Anti-Dictatorial Regime War
3. UN Security Council Authorized War
4. War for Oil
5. Racist War
6. Iraqi Threat to World Peace
7. Always Wrong War
8. War to Overthrow the Iraqi Regime
9. Feelings against Neoliberal Globalization
10. Governmental Dissatisfaction

Followed Computational Strategy

Step 1. Indexing countries as $\ell = 1, 2, \ldots, 8$, start with country $\ell = 1$.

Step 2. For any war attitude k, compute the projected 1-mode **weighted** graph $G_{B,k}(A)$ on the set of organizations, in which the weight on a pair of organizations is equal to the **number of respondents**, who are affiliated to both organizations and utter the **same war attitude** k.

Step 3. Detect the community partitions $\mathcal{C}_k = \mathcal{C}(G_{B,k}(A))$, for each war attitude k.

Step 4. Derive the **weighted** graph $G_{\mathcal{C}}(A)$ on the set of organizations, in which the weight on a pair of organizations is equal to the **number of partitions** $\mathcal{C}_k$'s such that both organizations belong to the **same community** in all these partitions.

Step 5. Detect the **meta–community** partition $\mathfrak{C}^{\ell} = \mathfrak{C}(G_{\mathcal{C}}(A))$ of the previous graph, for country $\ell = 1$.

Step 6. Repeat **Step 2.–5.**, for all countries.

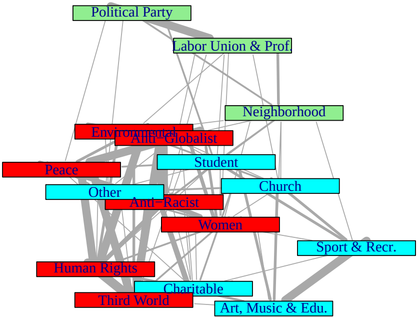
Step 7. From the family of 8 meta–community partitions, derive the **weighted** graph $G_{\mathfrak{C}}(A)$ on the set of organizations, in which the weight on a pair of organizations is equal to the **number of countries** such that both organizations belong to the **same meta–community** in all countries.

Step 8. Finally, detect the **meta–meta–community** partition $\mathfrak{C} = \mathfrak{C}(G_{\mathfrak{C}}(A))$ of the previous graph, which forms groups of organizations realtively strongly (frequently) clustered throughout all countries, when all war attitudes are taken into account.

ITALY: Membership of Organizations to Community ids for different War Attitudes ($N = 972$)

	USA Crusade against Islam	Anti-Dictatorial Regime War	UN Security Council Authorized War	War for Oil	Racist War	Iraqi Threat to World Peace	Always Wrong War	War to Overthrow the Iraqi Regime	Feelings against Neoliberal Globalization	Governmental Dissatisfaction
Church	7	4	1	8	6	5	3	1	5	3
Anti-Racist	6	1	4	2	2	2	7	2	1	3
Student	6	3	3	3	4	5	1	1	1	4
Labor Union – Prof.	4	2	3	1	7	1	6	6	4	3
Political Party	4	2	3	1	7	1	6	6	4	4
Women	1	1	2	4	5	2	8	2	1	3
Sport – Recr.	5	3	2	6	6	5	4	1	2	2
Environmental	6	1	4	5	2	6	9	2	1	3
Art, Music & Edu.	5	3	2	6	6	5	4	1	2	1
Neighborhood	3	3	3	9	1	7	2	6	3	3
Charitable	2	1	1	7	6	3	5	3	6	1
Anti-Globalist	8	1	4	2	3	4	7	4	1	3
Third World	8	1	4	2	3	4	7	4	1	1
Human Rights	8	1	4	2	3	4	7	4	1	1
Peace	8	1	4	2	3	4	7	4	1	4
Other	2	1	1	7	5	3	5	5	5	4

ITALY: Plot of the meta–community graph on organizations parametrized by all war attitudes



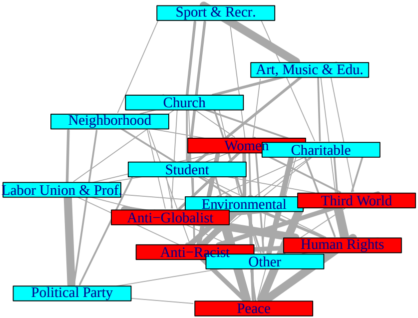
Three Meta–Communities:

1. Environmental, Anti–Globalist, Peace, Anti–Racist, Women, Human Rights, Third World
2. Student, Church, Sports & Recr., Charitable, Art – Music & Edu., Other
3. Political Party, Labor Union & Prof., Neighborhood

GERMANY: Membership of Organizations to Community ids for different War Attitudes ($N = 764$)

	USA Crusade against Islam	Anti-Dictatorial Regime War	UN Security Council Authorized War	War for Oil	Racist War	Iraqi Threat to World Peace	Always Wrong War	War to Overthrow the Iraqi Regime	Feelings against Neoliberal Globalization	Governmental Dissatisfaction
Church	1	2	3	2	3	1	10	7	4	7
Anti-Racist	2	3	4	1	6	2	4	2	5	2
Student	3	1	2	2	2	2	10	7	1	7
Labor Union – Prof.	7	2	4	3	4	6	1	5	2	1
Political Party	5	4	1	3	4	3	1	4	2	1
Women	2	4	1	9	1	2	3	2	5	9
Sport – Recr.	5	3	2	2	2	5	7	7	3	8
Environmental	4	3	2	6	7	2	5	1	1	4
Art, Music & Edu.	6	2	2	5	5	5	8	1	3	3
Neighborhood	1	2	4	4	5	5	9	6	3	9
Charitable	1	1	4	8	1	4	2	4	4	6
Anti-Globalist	2	4	1	1	6	2	4	2	5	2
Third World	2	4	1	1	6	2	4	2	5	2
Human Rights	2	4	1	1	6	2	4	2	5	2
Peace	2	4	1	1	6	2	4	2	5	2
Other	5	4	1	7	2	5	6	3	5	5

GERMANY: Plot of the meta–community graph on organizations parametrized by all war attitudes



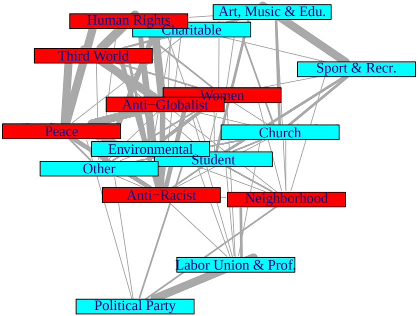
Two Meta–Communities:

1. Anti–Globalist, Peace, Anti–Racist, Women, Human Rights, Third World
2. Environmental, Student, Church, Sports & Recr., Charitable, Art – Music & Edu., Political Party, Labor Union & Prof., Neighborhood, Other

USA: Membership of Organizations to Community ids for different War Attitudes ($N = 662$)

	USA Crusade against Islam	Anti-Dictatorial Regime War	UN Security Council Authorized War	War for Oil	Racist War	Iraqi Threat to World Peace	Always Wrong War	War to Overthrow the Iraqi Regime	Feelings against Neoliberal Globalization	Governmental Dissatisfaction
Church	7	1	5	5	3	4	1	2	1	1
Anti-Racist	3	4	2	2	6	7	2	1	3	2
Student	2	2	3	4	2	1	3	5	5	2
Labor Union – Prof.	6	3	5	6	4	5	6	2	2	2
Political Party	6	4	1	6	4	5	6	3	2	1
Women	3	4	2	2	6	1	2	1	3	2
Sport – Recr.	7	1	5	1	2	2	4	6	5	2
Environmental	4	5	4	3	7	6	5	6	6	2
Art, Music & Edu.	5	5	1	1	2	4	1	2	6	2
Neighborhood	2	2	3	4	1	1	2	1	5	1
Charitable	1	5	1	7	7	3	5	2	6	1
Anti-Globalist	3	5	2	2	5	7	2	4	4	1
Third World	3	2	2	2	5	1	2	4	3	1
Human Rights	3	4	2	2	5	7	2	4	4	1
Peace	2	2	2	2	5	1	2	4	4	1
Other	5	4	1	7	2	3	3	3	5	2

USA: Plot of the meta–community graph on organizations parametrized by all war attitudes



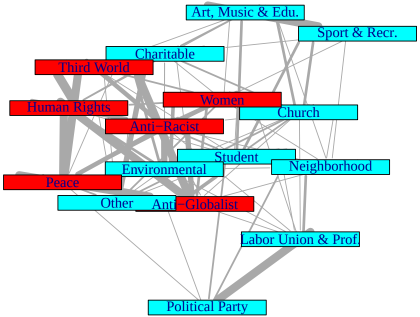
Two Meta–Communities:

1. Anti–Globalist, Peace, Anti–Racist, Women, Human Rights, Third World, Neighborhood
2. Environmental, Student, Church, Sports & Recr., Charitable, Art – Music & Edu., Political Party, Labor Union & Prof., Other

BELGIUM: Membership of Organizations to Community ids for different War Attitudes ($N = 644$)

	USA Crusade against Islam	Anti-Dictatorial Regime War	UN Security Council Authorized War	War for Oil	Racist War	Iraqi Threat to World Peace	Always Wrong War	War to Overthrow the Iraqi Regime	Feelings against Neoliberal Globalization	Governmental Dissatisfaction
Church	5	1	2	2	4	5	7	7	1	4
Anti-Racist	2	2	2	1	1	3	8	3	3	2
Student	4	1	4	8	3	3	1	2	4	1
Labor Union – Prof.	3	2	1	7	2	5	2	5	2	6
Political Party	4	1	4	6	3	2	6	4	1	5
Women	2	2	2	1	1	1	8	4	3	2
Sport – Recr.	4	1	4	5	3	1	5	1	4	1
Environmental	1	1	4	5	5	1	5	1	4	1
Art, Music & Edu.	6	1	4	4	4	1	3	7	5	7
Neighborhood	6	4	3	4	6	1	3	6	5	3
Charitable	6	4	1	3	4	1	4	6	5	3
Anti-Globalist	2	3	3	1	1	6	8	3	3	2
Third World	2	2	2	1	1	3	8	3	3	2
Human Rights	2	2	2	1	1	3	8	3	3	2
Peace	2	3	2	1	1	3	8	3	3	2
Other	4	2	1	6	3	4	6	5	4	5

BELGIUM: Plot of the meta–community graph on organizations parametrized by all war attitudes



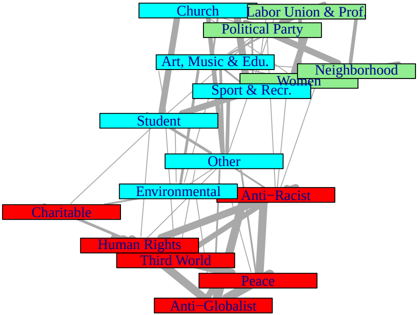
Two Meta–Communities:

1. Anti–Globalist, Peace, Anti–Racist, Women, Human Rights, Third World
2. Environmental, Student, Church, Sports & Recr., Charitable, Art – Music & Edu., Political Party, Labor Union & Prof., Neighborhood, Other

SWITZERLAND: Membership of Organizations to Community ids for different War Attitudes ($N = 740$)

	USA Crusade against Islam	Anti-Dictatorial Regime War	UN Security Council Authorized War	War for Oil	Racist War	Iraqi Threat to World Peace	Always Wrong War	War to Overthrow the Iraqi Regime	Feelings against Neoliberal Globalization	Governmental Dissatisfaction
Church	5	1	1	5	4	2	6	6	3	5
Anti-Racist	2	4	4	2	2	4	4	3	1	2
Student	5	2	1	5	4	3	6	6	4	5
Labor Union – Prof.	1	3	2	7	6	2	5	5	2	1
Political Party	3	3	2	3	4	4	5	5	2	1
Women	7	3	2	3	4	4	5	2	2	7
Sport – Recr.	5	1	2	5	4	3	6	6	3	5
Environmental	6	4	3	1	3	3	2	1	7	4
Art, Music & Edu.	6	1	2	1	1	2	1	1	4	6
Neighborhood	3	3	2	3	5	4	5	2	5	1
Charitable	4	2	3	4	3	5	3	4	6	3
Anti-Globalist	2	4	4	2	2	1	4	3	1	2
Third World	2	2	4	4	2	1	3	3	1	2
Human Rights	2	2	4	2	2	1	4	3	1	2
Peace	2	4	4	2	2	1	4	3	1	2
Other	5	4	4	6	1	2	6	6	3	6

SWITZERLAND: Plot of the meta–community graph on organizations parametrized by all war attitudes



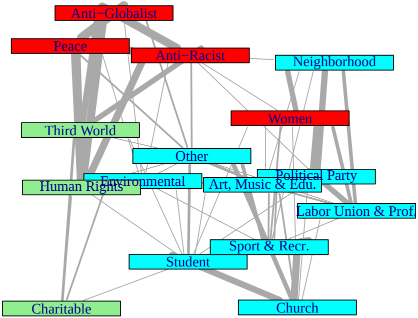
Three Meta–Communities:

1. Anti–Globalist, Peace, Anti–Racist, Human Rights, Third World, Charitable
2. Environmental, Student, Church, Sports & Recr., Art – Music & Edu., Other
3. Political Party, Labor Union & Prof., Neighborhood, Women

THE NETHERLANDS: Membership of Organizations to Community ids for different War Attitudes ($N = 605$)

	USA Crusade against Islam	Anti-Dictatorial Regime War	UN Security Council Authorized War	War for Oil	Racist War	Iraqi Threat to World Peace	Always Wrong War	War to Overthrow the Iraqi Regime	Feelings against Neoliberal Globalization	Governmental Dissatisfaction
Church	5	3	2	7	2	2	6	4	5	
Anti-Racist	1	1	6	1	2	2	7	3	1	
Student	2	4	4	2	3	6	2	1	3	
Labor Union – Prof.	2	3	3	5	3	6	3	5	4	
Political Party	2	3	3	2	3	6	2	6	3	
Women	1	3	6	1	2	3	7	3	1	
Sport – Recr.	4	4	4	6	1	4	4	7	5	
Environmental	6	5	5	2	5	1	1	1	7	
Art, Music & Edu.	3	2	5	8	1	4	3	5	5	
Neighborhood	3	5	6	4	2	2	2	6	3	
Charitable	7	1	6	9	2	2	8	2	6	
Anti-Globalist	1	1	6	1	2	3	7	3	1	
Third World	1	3	6	11	4	3	8	2	6	
Human Rights	1	5	7	10	4	3	8	2	6	
Peace	1	2	6	1	2	2	7	3	1	
Other	3	1	1	3	2	5	5	7	2	

THE NETHERLANDS: Plot of the meta–community graph on organizations parametrized by all war attitudes



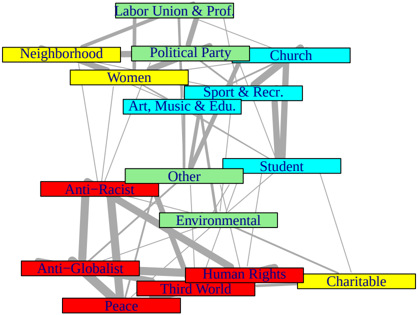
Three Meta–Communities:

1. Anti–Globalist, Peace, Anti–Racist, Women, Other
2. Environmental, Student, Labor Union & Prof., Political Party, Church, Sports & Recr., Art – Music & Edu., Neighborhood
3. Charitable, Human Rights, Third World

UK: Membership of Organizations to Community ids for different War Attitudes ($N = 496$)

	USA Crusade against Islam	Anti-Dictatorial Regime War	UN Security Council Authorized War	War for Oil	Racist War	Iraqi Threat to World Peace	Always Wrong War	War to Overthrow the Iraqi Regime	Feelings against Neoliberal Globalization	Governmental Dissatisfaction
Church	1	5	2	1	6	3	3	3	5	
Anti-Racist	4	5	1	2	3	1	1	1	5	
Student	4	5	2	2	4	3	6	3	5	
Labor Union – Prof.	2	1	1	6	3	2	7	5	1	
Political Party	2	4	6	6	3	2	7	5	1	
Women	3	2	6	4	3	1	4	6	1	
Sport – Recr.	5	5	2	7	2	3	6	3	6	
Environmental	2	3	4	6	3	4	7	4	1	
Art, Music & Edu.	6	4	5	1	5	3	6	3	3	
Neighborhood	8	3	1	3	1	1	4	6	5	
Charitable	8	2	3	5	6	4	5	2	4	
Anti-Globalist	7	1	1	2	4	1	2	2	2	
Third World	7	1	1	5	4	1	2	2	2	
Human Rights	7	1	4	2	4	1	2	2	2	
Peace	4	3	4	2	4	1	1	1	2	
Other	8	2	6	3	1	1	5	6	4	

UK: Plot of the meta–community graph on organizations parametrized by all war attitudes



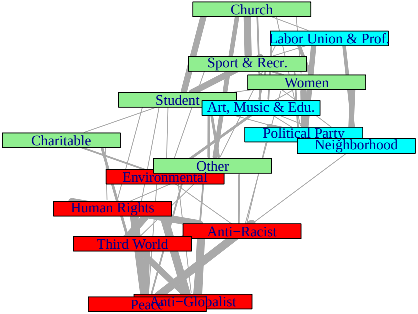
Four Meta–Communities:

1. Anti–Globalist, Peace, Anti–Racist, Human Rights, Third World
2. Student, Church, Sports & Recr., Art – Music & Edu.
3. Environmental, Political Party, Labor Union & Prof., Other
4. Women, Neighborhood, Charitable

SPAIN: Membership of Organizations to Community ids for different War Attitudes ($N = 452$)

	USA Crusade against Islam	Anti-Dictatorial Regime War	UN Security Council Authorized War	War for Oil	Racist War	Iraqi Threat to World Peace	Always Wrong War	War to Overthrow the Iraqi Regime	Feelings against Neoliberal Globalization	Governmental Dissatisfaction
Church	1	1	2	4	2	5	5	6	1	1
Anti-Racist	2	2	3	6	1	6	4	2	4	1
Student	1	1	2	4	3	2	6	6	2	1
Labor Union – Prof.	5	3	4	3	3	3	1	5	3	1
Political Party	5	3	1	3	5	3	1	5	3	1
Women	1	1	2	1	1	2	2	3	2	1
Sport – Recr.	1	4	4	2	2	2	6	1	2	1
Environmental	2	3	3	1	6	6	2	2	5	1
Art, Music & Edu.	5	4	4	3	4	3	1	5	3	1
Neighborhood	5	4	2	3	2	4	1	5	3	1
Charitable	4	2	2	6	6	1	4	3	5	1
Anti-Globalist	2	3	3	1	1	6	4	2	4	1
Third World	3	2	3	1	1	6	2	4	4	1
Human Rights	2	2	3	1	1	6	2	2	4	1
Peace	2	1	2	1	1	6	2	2	4	1
Other	4	1	2	5	2	1	3	6	1	1

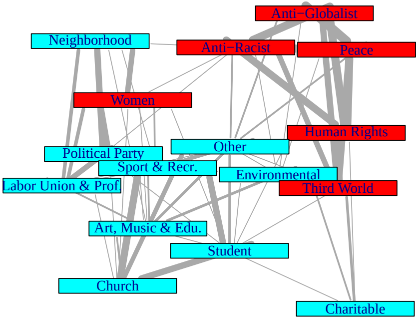
SPAIN: Plot of the meta–community graph on organizations parametrized by all war attitudes



Three Meta–Communities:

1. Environmental, Anti–Globalist, Peace, Anti–Racist, Human Rights, Third World
2. Student, Women, Church, Sports & Recr., Charitable, Other
3. Political Party, Labor Union & Prof., Neighborhood, Art – Music & Edu.

ALL 8 COUNTRIES: Plot of the Meta–Meta–Community Graph on Organizations Parametrized by All War Attitudes



Two Meta–Meta–Communities:

1. Environmental, Student, Church, Sports & Recr., Charitable, Art – Music & Edu., Political Party, Labor Union & Prof., Neighborhood, Other
2. Anti–Globalist, Peace, Anti–Racist, Human Rights, Third World, Women

Communities in Graphs

Let G be a graph on a set of vertices V.

A **community structure** in G is a partition of V in a family of subsets $\mathcal{C} = \mathcal{C}(V) = \{C_1, C_2, \ldots, C_p\}$, called **communities**, such that $\mathcal{C}$ maximizes the following benefit function Q, called **modularity**, which is defined (Newman & Girvan [2004]) as:

$$Q = \text{(fraction of connections within communities)} \\ \text{- (expected fraction of such connections).}$$

In the **null model**, the expected fraction above is calculated on the basis of a random graph, which preserves the same degree distribution with the examined graph G.

Thus, the exact expression of modularity becomes:

$$Q = \frac{1}{2m} \sum_{i,j \in V} (A_{ij} - \frac{d_i d_j}{2m})\ \delta(C_i, C_j),$$

where

- A_{ij} is the adjacency matrix of G,
- m the total number of connections in G,
- d_i is the degree of vertex i,
- C_i denotes the community in $\mathcal{C}$, to which vertex i belongs, and
- the δ-function is defined as $\delta(C_i, C_j) = 1$, whenever i and j belong to the same community, and $\delta(C_i, C_j) = 0$, otherwise.

Properties of Q:

- By normalization in definition, $-1 \leq Q \leq 1$.
- $Q = 0$ if and only if the whole graph is a single community (i.e., $|\mathcal{C}| = 1$).
- If every vertex of the graph is a community–singleton (i.e., $|\mathcal{C}| = |V|$), then $Q \leq 0$.
- If $Q \leq 0$, for every partition, then G has no community structure (in fact, such a graph would be strongly *multipartite-like*, in the sense that it would be decomposed to subgraphs with very few internal connections and many external connections between them).

Modularity Maximization

- If $\max Q > 0$, over *all* possible partitions, the graph has a community structure, in the sense that most of the graph connections fall within the communities (of the optimal partition) than what would have been expected by chance (under the null model). This community structure is stronger the more Q approaches to 1.
- However, this optimization problem has been proven to be NP-complete (Brandeis et al., 2008) and, thus, only approximate optimization techniques, such as greedy algorithms, simulated annealing, extremal optimization, expectation maximization, spectral methods etc. can be practically useful.

Constructing Meta–Communities

- Let us denote by $\mathcal{X} = \mathcal{X}(V)$ the set of graphs, all having the same set of vertices V.
- Let $\mathcal{H} = \mathcal{H}(V)$ be an arbitrary (non-trivial) family of such graphs, i.e., $\mathcal{H} \in 2^{\mathcal{X}}$.
- Assume that, for any $G \in \mathcal{X}$, we are able to determine a community partitioning $\mathcal{C}(G)$. (In fact, our argument is valid for any sort of graph clustering.)

Then, given the family $\mathcal{H}$, for any $u, v \in V$, we define a non-negative integer $w(u, v)$ as follows:

$$w(u, v) =$$

$$\#\{G \in \mathcal{H} \colon u \text{ and } v \text{ belong to the same community in } \mathcal{C}(G)\}.$$

This relation (on $V \times V$) constructs a weighted graph (on V), for any family $\mathcal{H}$ of graphs (on V). Therefore, there exists a mapping

$$\varphi \colon 2^{\mathcal{X}} \to \mathcal{X},$$

which associates to any family of graphs a graph that measures how strong (in terms of frequency) is the community binding in V throughout all graphs of this family. The weighted graph $\varphi(\mathcal{H})$ (on V) will be called "**community bonder graph**" of the family $\mathcal{H}$ of graphs (on V).

Nonetheless, the graph $\varphi(\mathcal{H})$ might be partitioned too into a set of **meta–communities** of vertices of V, indicating how much relatively persistent is the membership to the original communities throughout the whole family $\mathcal{H}$. In fact, two vertices belong to a meta-community, whenever they both belong to relatively more communities formed in $\mathcal{H}$ than vertices lying across different meta–communities.

Furthermore, this process can continue for any families of families of ... of families of graphs, because (through φ) $2^{2^{\cdot^{\cdot^{\cdot^{2^{\mathcal{X}}}}}}}$ is alway isomorphic to $\mathcal{X}$.

APPENDIX: Partial Tripartite Graphs

A graph is said **tripartite** (or **3-mode**) if the set of its vertices is decomposed in three disjoint sets, called **modes**, such that no two vertices within the same set (mode) are adjacent.

Apparently this is a direct generalization of the notion of **bipartite** (or **2-mode**) graphs.

A graph is said **partial tripartite** if it is tripartite, but no two vertices within two of its modes are adjacent.

From now on, a graph G with vertex set V will be denoted as $G(V)$ and we will use a similar notation for multipartite graphs too.

For three disjoint (non-empty) sets A, B and C, we denote a tripartite graph as $G(A, B, C)$, where these three sets are its modes, but no two vertices within A and C are adjacent.

In $G(A, B, C)$, B is the **middle** mode and A and B are the **edging** modes. Note the symmetry (up to a permutation of vertices):

$$G(A, B, C) = G(C, B, A).$$

Similarly, for two disjoint (non-empty) sets X and Y, we denote a bipartite graph as $G(X, Y)$. Note the symmetry (up to permutation):

$$G(X, Y) = G(Y, X).$$

In the partial tripartite graph $G(A, B, C)$, for any $p \in A, q \in B$ and $r \in C$, one may define four polar sets:

- the **right-polar** set of p, denoted as $p' \subset B$, is the set of all $j \in B$, which are connected with p,
- the **left-polar** set of q, denoted as $'q \subset A$, is the set of all $i \in A$, which are connected with q,
- the **right-polar** set of q, denoted as $q' \subset C$, is the set of all $k \in C$, which are connected with q and
- the **left-polar** set of r, denoted as $'r \subset B$, is the set of all $j \in B$, which are connected with r.

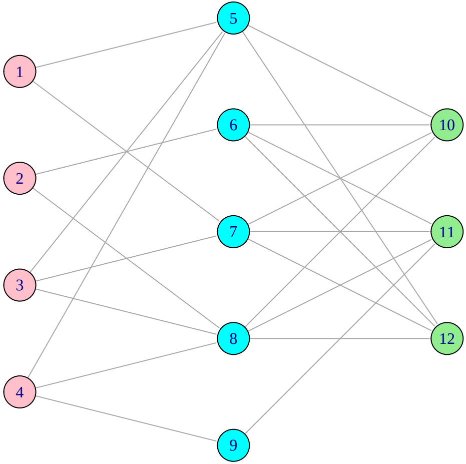
Note that, denoting by d_s the **degree** of any vertex s in the graph $G(A, B, C)$,

- $d_i = |i'|$, for any $i \in A$,
- $d_j = |'j| + |j'|$, for any $j \in B$,
- $d_k = |'k|$, for any $k \in C$.

A partial tripartite graph $G(A, B, C)$ is said (**strongly**) **connected** if vertices are connected to each other across the three modes as follows:

- for any $i \in A$, there exists a $j \in B$, such that $i \in {}'j$,
- for any $j \in B$, there exists both a $i \in A$, such that $j \in i'$, and a $k \in C$, such that $j \in {}'k$ and
- for any $k \in C$, there exists a $j \in B$, such that $k \in j'$.

Example: The plot of a small partial tripartite graph with $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$ and $C = \{10, 11, 12\}$.



Reducing Partial Tripartite Graphs to Bipartite Graphs

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their union is defined as the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Any connected partial tripartite graph $G(A, B, C)$ can be reduced to one or the union of more bipartite graphs as follows:

I. $G(A, B, C) = G([A, C], B)$,

II. $G(A, B, C) = G(A, B) \cup G(B, C)$,

III. $G(A, B, C) = \bigcup_{i \in A} G(i, B, C) = \bigcup_{j \in B} G(A, j, C) = \bigcup_{k \in C} G(A, B, k)$,

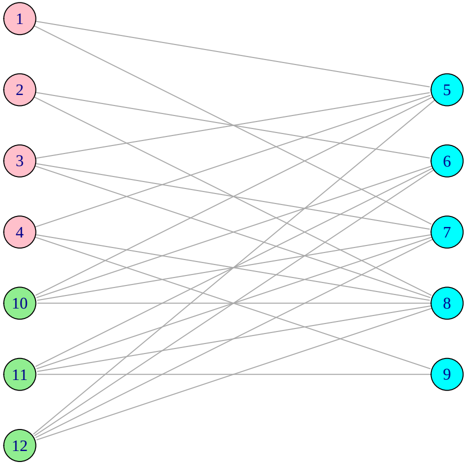
where $[A, C]$ denotes the concatenation of modes A and C and $G(A, B, k)$ is the partial tripartite graph resulting when removing from $G(A, B, C)$ all vertices $j \notin {}'k$ and all vertices $i \notin {}'({}'k)$ (and similarly for $G(A, j, C)$ and $G(i, B, C)$).

Rather problematic cases:

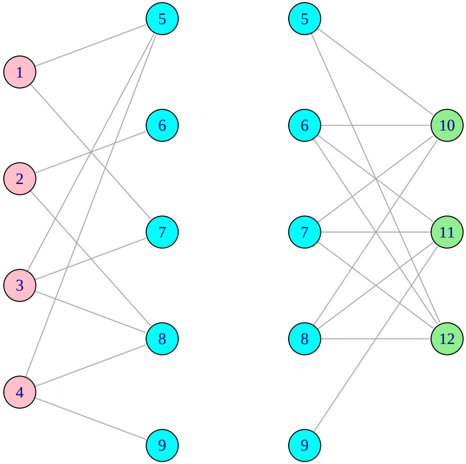
I. The problem with such a reduction is that it **lumps together** modes A and C. However, in empirical data, these modes might correspond to completely different and immiscible objects.

II. Now the problem is that any analysis of one of the two bipartite components would be **independent** of the other. However, in empirical data, modes are dependent to each other and so the analysis should be irreducible.

The lumped bipartite form $G([A, C], B)$ of the partial tripartite $G(A, B, C)$:



The two separated bipartite components $G(A, B)$ and $G(B, C)$ of the partial tripartite $G(A, B, C)$:

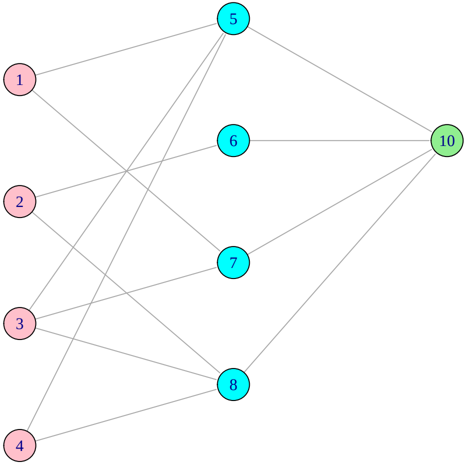


A viable scenario:

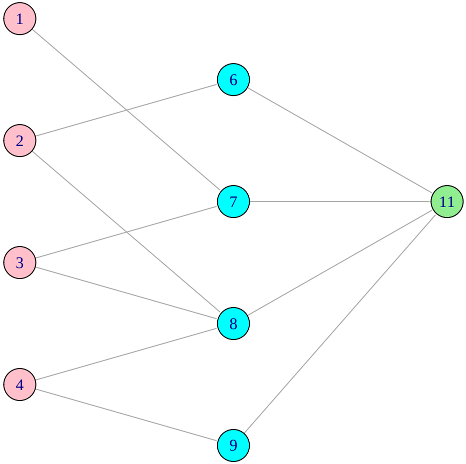
III. As a matter of fact, in most empirical cases, say, mode C stands for **attributes** of vertices of mode B. Since in $G(A, B, k)$, *every* vertex of B is connected to the same $k \in C$, one may say that $G(A, B, k)$ is an **attributably homogeneous** graph. Furthermore, the bipartite component $G(B, k)$ (of $G(A, B, k)$) essentially can be seen as a **parametrization** B_k of the set B, for a (parameter) $k \in C$. Thus, through such parametrization, the partial tripartite graph $G(A, B, k)$ may be reduced to the bipartite graph $G(A, B_k)$. Of course, for any $k \in C$, we have that $B_k = \{'k\} \subseteq B$ and this means that $G(A, B_k)$ can be identified with a bipartite graph $G_k(A, B)$, which is defined as a modification of $G(A, B)$ by removing any connections of A with $B \setminus \{'k\}$. In this way, we may write:

$$G(A, B, k) = G_k(A, B).$$

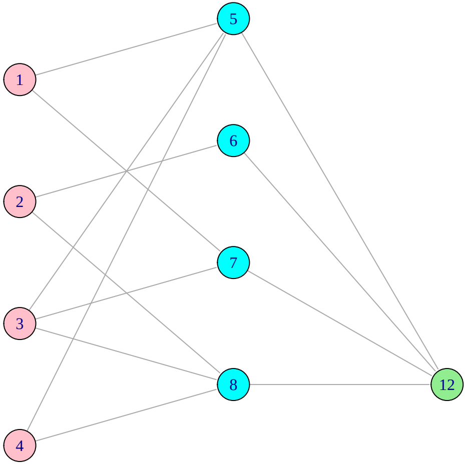
The attributably homogeneous bipartite component $G(A, B, 10)$ of the partial tripartite $G(A, B, C)$:



The attributably homogeneous bipartite component $G(A, B, 11)$ of the partial tripartite $G(A, B, C)$:



The attributably homogeneous bipartite component $G(A, B, 12)$ of the partial tripartite $G(A, B, C)$:



Incidence Matrices (I. and II.)

The **incidence matrices** corresponding to the three representations of the partial tripartite graph as bipartite graph(s) are the following:

I. The incidence matrix $M(B, [A, C])$ (of order $|B| \times (|A| + |C|)$) such that $M_{ij}(B, [A, C]) = 1$, whenever the vertices $i \in B$ and $j \in A \cup C$ are connected, and $M_{ij}(B, [A, C]) = 0$, otherwise.

II. The incidence matrix $M(A, B)$ (of order $|A| \times |B|$) such that $M_{ij}(A, B) = 1$, whenever the vertices $i \in A$ and $j \in B$ are connected, and $M_{ij}(A, B) = 0$, otherwise. (Similarly, for the incidence matrix $M(B, C)$.)

Incidence Matrices (III.)

III. For any (fixed) $k \in C$, the incidence matrix $M(A, B, k)$ (of order $|A| \times |B|$) such that $M_{ij}(A, B, k) = 1$, whenever $i \in A$ is connected to $j \in B$, which is connected to $k \in C$, and $M_{ij}(A, B, k) = 0$, otherwise. (Similarly, for the incidence matrix $M(A, j, C)$ and $M(i, B, C)$.)

Note that here we have identified the tripartite $G(A, B, k)$ with the bipartite $G_k(A, B)$, as we have already explained (and this will be done implicitly from now on in any such case). Otherwise, the order of $M(A, B, k)$ would be $|A| \times (|B| + 1)$.

Properties of these incidence matrices:

- $M(B,[A,C]) = M([A,C],B)^{\mathsf{T}}$.
- $M(B,A)$ and $M(B,C)$ are two blocks in $M(B,[A,C])$:

$$M(B,[A,C]) = [M(B,A) \mid M(B,C)].$$

- $M(B,A) = M(A,B)^{\mathsf{T}}$ (Similarly, for $M(B,C)$.)
- $M(B,A,k) = M(A,B,k)^{\mathsf{T}}$ (Similarly, for $M(i,B,C)$ and $M(A,j,C)$.)
- For any (fixed) $k \in C$, $M(A,B,k)$ is equal to what $M(A,B)$ would be by setting 0 all the entries of the j-th column, for any $j \notin {}'k$. (Similarly, for $M(A,j,C)$ and $M(i,B,C)$.)

Projections to 1-Mode Graphs

Correspondingly, to the bipartite representation of a partial tripartite graph $G(A, B, C)$ we have the following projections to the following 1-mode **weighted** (undirected) graphs (here we refer just to some of the projections, but similarly one could construct the others too):

I. $G_{[A,C]}(B)$ denotes the projection of $G(B, [A, C])$ over B. In this graph, two vertices $i, j \in B$ are connected if there exists at least one $k \in A \cup B$, which is connected to both i and j, i.e., it is such that $M_{ik}(B, [A, C]) = M_{jk}(B, [A, C]) = 1$. The weight of this connection $w_{ij}(B|[A, C])$ equals the total number of such k's (zero, if none).

II. $G_B(A)$ denotes the projection of $G(A, B)$ over A. In this graph, two vertices $i, j \in A$ are connected if there exists at least one $p \in B$, which is connected to both i and j, i.e., it is such that $M_{ip}(A, B) = M_{jp}(A, B) = 1$. The weight of this connection $w_{ij}(A|B)$ equals the total number of such p's (zero, if none).

III. For any $k \in C$, $G_{B,k}(A)$ denotes the projection of $(G(A, B, k) =)$ $G(A, B_k)$ over A. In this graph, two vertices $i, j \in A$ are connected if there exists at least one $q \in B$, which is connected to both i, j and to k too, i.e., it is such that $M_{iq}(A, B, k) = M_{jq}(A, B, k)$ $= 1$. The weight of this connection $w_{ij}(A|B, k)$ equals the total number of such k's (zero, if none).

Adjacency Matrices

The **adjacency matrices** of the above projections are derived from the corresponding incidence matrices as follows:

I. $\text{AdjacencyMatrix}(G_{[A,C]}(B)) = M(B,[A,C])\ M(B,[A,C])^{\mathsf{T}} = M(B,[A,C])\ M([A,C],B)$.

II. $\text{AdjacencyMatrix}(G_B(A)) = M(A,B)\ M(A,B)^{\mathsf{T}} = M(A,B)\ M(B,A)$.

III. For any $k \in C$, $\text{AdjacencyMatrix}(G_{B,k}(A)) = M(A,B,k)\ M(A,B,k)^{\mathsf{T}} = M(A,B,k)\ M(B,A,k)$.

Properties of the adjacency matrices:

- $\mathrm{AdjacencyMatrix}(G_{[A,C]}(B)) = \mathrm{AdjacencyMatrix}(G_A(B)) + \mathrm{AdjacencyMatrix}(G_C(B))$.
- For any (fixed) $k \in C$, $\mathrm{AdjacencyMatrix}(G_{B,k}(A))$ is equal to what $\mathrm{AdjacencyMatrix}(G_B(A))$ would be by setting 0 all the entries of the j-th rows and columns, for any $j \notin {}'k$.

Matrix Computations in the Example

In our example, we have the following incidence matrices:

$$M(B,[A,C]) = [M(B,A) \mid M(B,C)] = \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right],$$

$$M(A,B,10) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, M(A,B,11) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$M(A,B,12) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In this example, the adjacency matrix of the projection on A, $G_B(A)$, is:

$$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix},$$

while the adjacency matrices of the three projections on B, $G_{[A,C]}(B)$, $G_A(B), G_C(B)$, are, respectively:

$$\begin{bmatrix} 5 & 2 & 4 & 4 & 1 \\ 2 & 4 & 3 & 4 & 1 \\ 4 & 3 & 5 & 4 & 1 \\ 4 & 4 & 4 & 6 & 2 \\ 1 & 1 & 1 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 2 & 1 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 2 & 2 & 0 \\ 2 & 3 & 3 & 3 & 1 \\ 2 & 3 & 3 & 3 & 1 \\ 2 & 3 & 3 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

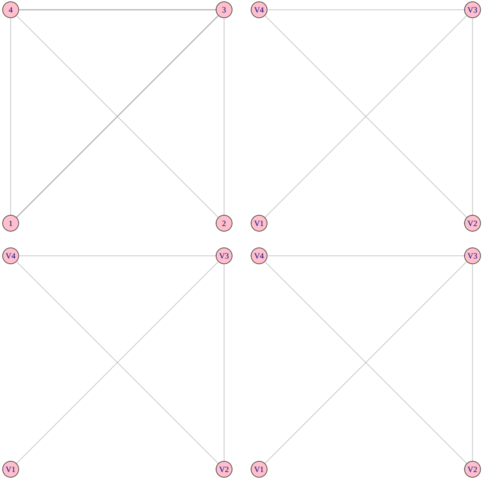
Moreover, for the three projections on A parametrized by C, the adjacency matrices of $G_{B,10}(A)$, $G_{B,11}(A)$, $G_{B,12}(A)$ are, respectively:

$$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix},$$

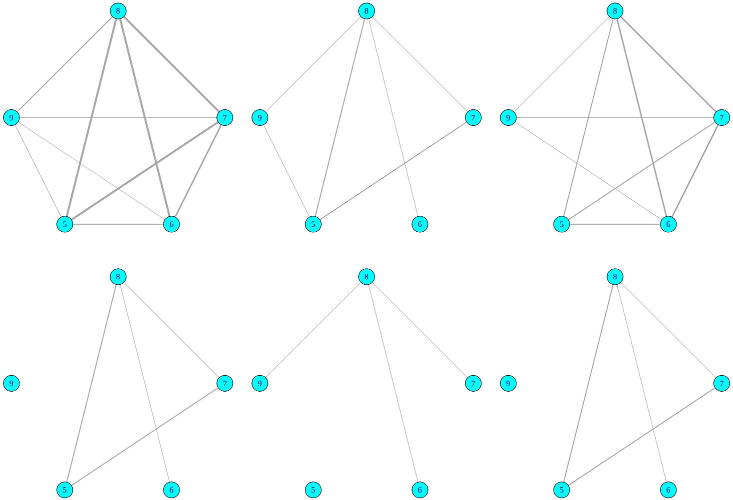
while, for the three projections on B parametrized by C, the adjacency matrices of $G_{A,10}(B)$, $G_{A,11}(B)$, $G_{A,12}(B)$ are, respectively:

$$\begin{bmatrix} 3 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 2 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 2 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Plots of the graphs $G_B(A)$, $G_{B,10}(A)$, $G_{B,11}(A)$, $G_{B,12}(A)$, respectively, of the example:



Plots of the graphs $G_{[A,C]}(B)$, $G_A(B)$, $G_C(B)$, $G_{A,10}(B)$, $G_{A,11}(B)$, $G_{A,12}(B)$, respectively, of the example:



Communities in the Example

In our example, corresponding to graphs $G_{[A,C]}(B)$, $G_B(A)$, $G_A(B)$, $G_C(B)$, $G_{B,10}(A)$, $G_{B,11}(A)$, $G_{B,12}(A)$, $G_{A,10}(B)$, $G_{A,11}(B)$, $G_{A,12}(B)$, we find the following sets of communities (for the vertices of B),respectively:

$$\begin{gathered}
\mathcal{C}_{[A,C]}(B) = \{2,1,2,1,3\}, \\
\mathcal{C}_{[B]}(A) = \{2,3,2,1\}, \\
\mathcal{C}_A(B) = \{2,3,2,3,1\}, \\
\mathcal{C}_C(B) = \{1,1,1,1,2\}, \\
\mathcal{C}_{B,10}(A) = \{1,3,1,2\}, \\
\mathcal{C}_{B,11}(A) = \{1,3,1,2\}, \\
\mathcal{C}_{B,12}(A) = \{1,3,1,2\}, \\
\mathcal{C}_{A,10}(B) = \{2,1,2,1,1\}, \\
\mathcal{C}_{A,11}(B) = \{2,2,1,3,3\}, \\
\mathcal{C}_{A,12}(B) = \{2,1,2,1,1\}.
\end{gathered}$$

Furthermore, the two community bonder (weighted) graphs $\mathcal{G}_{B,C}(A), \mathcal{G}_{A,C}(B)$ have adjacency matrices, respectively:

$$\begin{bmatrix} 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 \\ 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix},$$

and, thus, the corresponding set of meta–communities is:

$$\mathfrak{C}_{B,C}(A) = \{1, 1, 1, 1\},$$

$$\mathfrak{C}_{A,C}(B) = \{2, 1, 2, 1, 1\}.$$